Lecture 11

Note Title 5/21/2009

Potential vorticity in a continuously stratified ocean → the Ertel PV

- While layered models are useful for conceptualizing some key concepts in the fluid dynamics of the ocean, they are deficient in that they cannot truly capture:

  (1) The continuous nature of the density and velocity fields that have:

  $$\frac{\partial \vec{u}}{\partial z}, \frac{\partial \rho}{\partial z}$$ vertical gradients that are not just confined to the interfaces.

- For a continuously stratified fluid, like the layered system one can derive a scalar quantity that is conserved in the absence of friction and diabatic processes (I'll define this in a minute) and that can tell you many things about the key dynamical variables of the system $(\vec{u}, \vec{\omega}_a, \rho)$.

This scalar quantity is the Ertel PV, which for a Boussinesq fluid is:

$$\boxed{\begin{gathered} q = \vec{\omega}_a \cdot \nabla b \\ \vec{\omega}_a = f\hat{k} + \nabla \times \vec{u} \qquad b = -\frac{g\rho}{\rho_0} \end{gathered}}$$

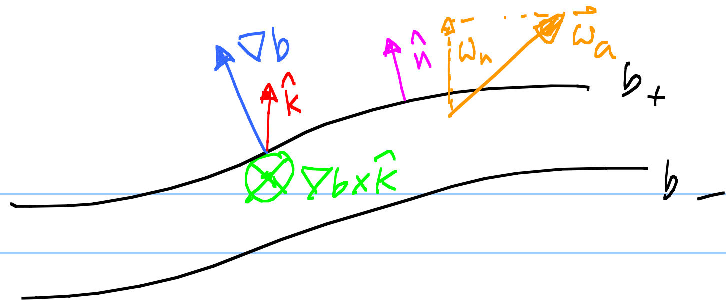
What does this quantity represent and why is it conserved for an inviscid, adiabatic fluid?

The vorticity equation in a baroclinic fluid you will recall is

$$\frac{D\vec{\omega}_a}{Dt} = \underbrace{\vec{\omega}_a \cdot \nabla \vec{u}}_{\text{vortex tilting squashing / stretching}} + \underbrace{\nabla b \times \hat{k}}_{\text{baroclinic torque}} + \underbrace{\nabla \times \vec{F}}_{\text{frictional torque}}$$

Which you will see implies that even in an inviscid fluid $\vec{\omega}_a$ changes following fluid parcels because of vortex tilting / squashing / stretching and the baroclinic torque.

Now notice the direction that the baroclinic torque is relative to density surfaces:



$$\hat{n} = \frac{\nabla b}{|\nabla b|}$$

The baroclinic torque will generate vorticity that is PARALLEL to density surfaces it will NOT affect the component of the absolute vorticity NORMAL to isopycnals.

$$\omega_n = \vec{\omega}_a \cdot \hat{n} = \omega_a \cdot \frac{\nabla b}{|\nabla b|}$$

Since

$$\hat{n} \cdot \left(\nabla b \times \hat{k} \right) = \frac{\nabla b}{|\nabla b|} \cdot \left(\nabla b \times \hat{k} \right) = 0$$

So you can see that the Ertel PV is proportional to the component of the vorticity normal to isopycnals

$$q = |\nabla b| \, \omega_n$$

and thus is unaffected by the baroclinic torque.

What about the vortex tilting/stretching term?

We can show that this term does not affect the PV by deriving an evolution equation for the PV.

$$\frac{Dq}{Dt} = \frac{D}{Dt}\left(\vec{\omega}_a \cdot \nabla b\right) = \nabla b \cdot \frac{D\vec{\omega}_a}{Dt} + \vec{\omega}_a \cdot \frac{D}{Dt} \nabla b$$

$$\nabla b \cdot \left(\frac{D\vec{\omega}_a}{Dt} \right) = \nabla b \cdot \left(\vec{\omega}_a \cdot \nabla \vec{u} + \nabla b \times \hat{k} + \nabla \times \vec{F} \right)$$

$$= \nabla b \cdot \vec{\omega}_a \cdot \nabla \vec{u} + \nabla b \cdot \nabla \times \vec{F}$$

What governs the evolution of the buoyancy gradient, what is $D\nabla b/Dt$?

$$\frac{Db}{Dt} = \mathcal{D} = \text{diabatic processes}$$

For example, assuming a linear equation of state:

$$\rho = \rho_0 \left[-\alpha \underbrace{(T - T_0)}_{T'} + \beta \underbrace{(S - S_0)}_{S'} \right]$$

$$b = -\frac{g\rho}{\rho_0} = \alpha g T' - \beta g S'$$

And if
$$\frac{DT'}{Dt} = K_T \nabla^2 T'$$

$$\frac{DS'}{Dt} = K_S \nabla^2 S'$$

$$\frac{Db}{Dt} = \alpha g K_T \nabla^2 T' - \beta g K_S \nabla^2 S'$$

$$\mathcal{D} = \alpha g K_T \nabla^2 T' - \beta g K_S \nabla^2 S'$$

related to small scale mixing of temperature and salinity

From the buoyancy equation we can calculate an equation for the buoyancy gradient by taking the gradient of that equation:

$$\nabla\left(\frac{Db}{Dt}\right) = \nabla\mathcal{D}$$

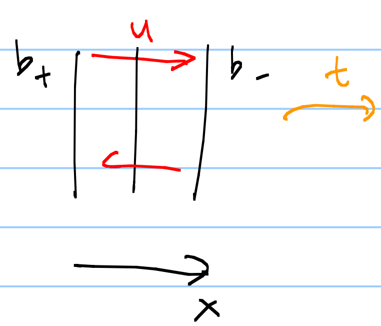
$$\nabla\left(\frac{\partial b}{\partial t} + \vec{u}\cdot\nabla b\right) = \frac{D}{Dt}\nabla b + \nabla b\cdot\nabla\vec{u}$$

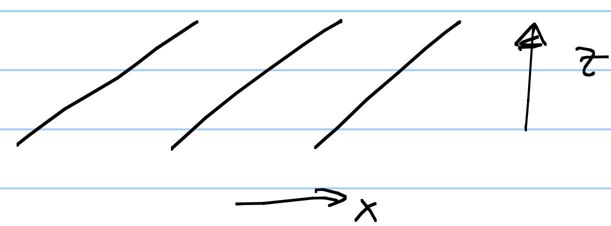
$$\frac{D}{Dt}\nabla b = -\underbrace{\nabla b\cdot\nabla\vec{u}} + \nabla\mathcal{D}$$

Generation of buoyancy gradients by the tilting and squeezing and straining of isopycnals

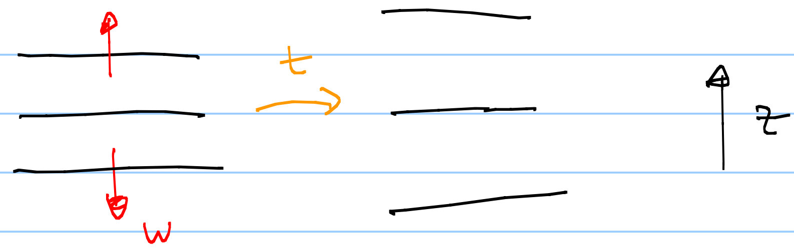
Take for example the vertical component of this equation:

$$\frac{D}{Dt}\left(\frac{\partial b}{\partial z}\right) = -\frac{\partial b}{\partial x}\frac{\partial u}{\partial z} - \frac{\partial b}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial b}{\partial z}\frac{\partial w}{\partial z} + \frac{\partial\mathcal{D}}{\partial z}$$





Tilting over of the isopycnals by the sheared zonal flow generates $\partial b/\partial z > 0 \rightarrow$ stratification



The straining of isopycnals by the vertical motion w/ $\partial w/\partial z > 0$ reduces the stratification $\partial b/\partial z \downarrow$.

$$\frac{Dq}{Dt} = \nabla b \cdot \frac{D\vec{\omega}_a}{Dt} + \vec{\omega}_a \cdot \frac{D\nabla b}{Dt}$$

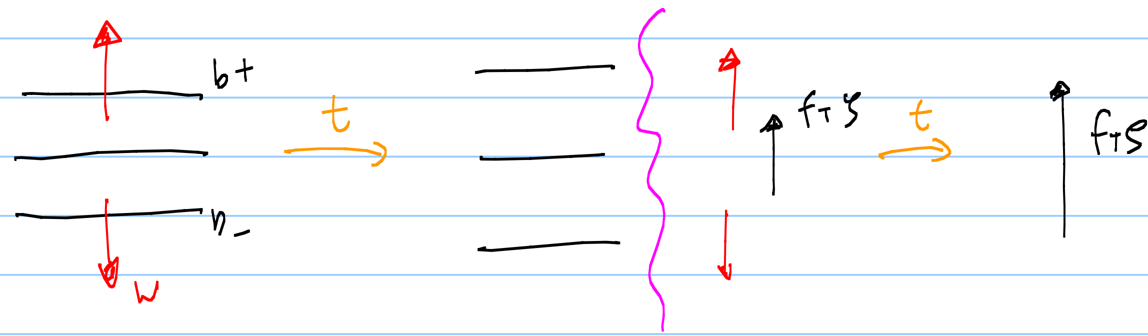
$$= \nabla b \cdot \Big(\underbrace{\cancel{\vec{\omega}_a \cdot \nabla \vec{u}}}_{\text{VORTEX TILT/STRETCH/SQUASH}} + \nabla \times \vec{F} \Big) + \cancel{\vec{\omega}_a} \cdot \Big(\underbrace{\cancel{-\nabla b \cdot \nabla \vec{u}}}_{\text{ISOPYCNAL TILTING/STRAINING}} + \nabla D \Big)$$

CANCEL
& thus do not affect the PV

$$\Rightarrow \boxed{\frac{Dq}{Dt} = \nabla b \cdot \nabla \times \vec{F} + \vec{\omega}_a \cdot \nabla D}$$

Only frictional & diabatic processes change the PV, vortex tilting (squashing/stretching and the tilting and straining of isopycnals which are purely advective processes do not affect the PV.

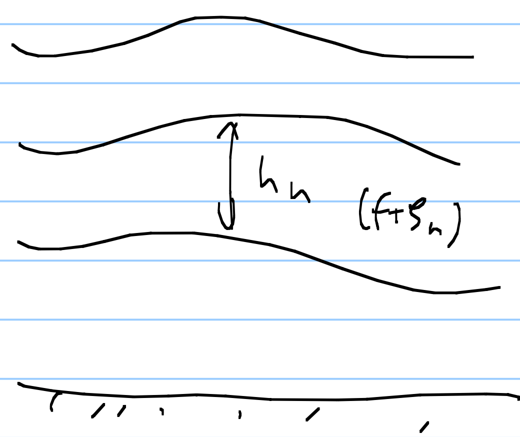
Why is this? Take the following example:



① $\partial w/\partial z > 0$ reduces the stratification but also → ② stretches the vertical component of the absolute vorticity

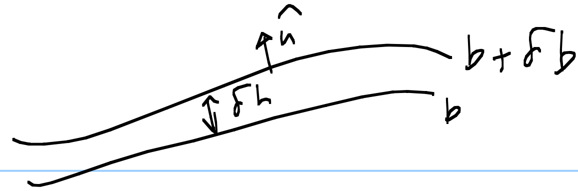
→ the product of the two quantities $(f+\zeta)\frac{\partial b}{\partial z}$ doesn't change in time

This is analogous to the result in the layered system



If h_n increased then $f+\zeta_n$ must increase to conserve PV $q_n = \frac{(f+\zeta_n)}{h_n}$

The connection between the PV in the layered and continuously stratified systems can be made even clearer by realizing that in the limit of an infinitesimally small distance in between isopycnals δh



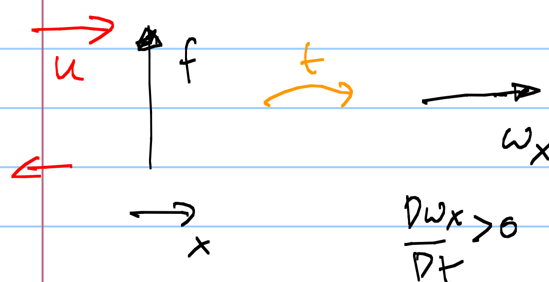
Then $\nabla b = \frac{\delta b}{\delta h}\hat{n}$

So that $q = \frac{\vec{\omega}_a \cdot \hat{n}\, \delta b}{\delta h} = \frac{\omega_n\, \delta b}{\delta h}$

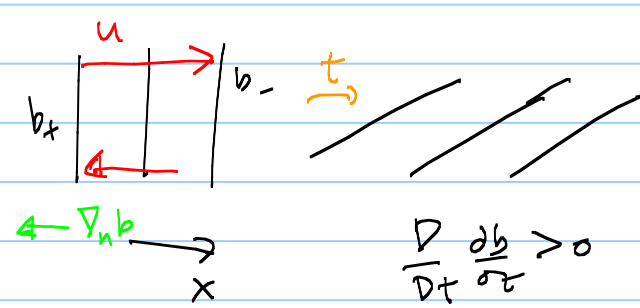
Which you can see takes a similar form to the layered system version of the PV $q_n = \frac{f+\zeta_n}{h_n}$

differing by a factor of δb.

There is another difference between PV conservation in the layered and continuous systems. In the continuous system the flow can have vertical shear (that is not only confined to the layer interfaces). Thus vortex tilting can arise, e.g.



x-component of vorticity increases



If there is $\partial b/\partial x < 0$ sheared flow will create stratification

$$q = fN^2 + \omega_x \frac{\partial b}{\partial x} < 0$$

Tilting of isopycnals increases stratification

Tilting of vorticity to the horizontal increase ω_x which makes this term more negative and the two effects cancel.

Adiabatic inviscid processes do not affect the PV following fluid parcels. In the absence of friction & diabatic processes the Ertel PV is conserved

$$\frac{Dq}{Dt} = 0 \qquad \text{No friction / diabatic processes}$$

Thus if a fluid parcel is instilled with a particular value of the PV, it will retain that value along the path of its motion.

In this way the PV is a tracer that tags or identifies water, and is thus like a dye. Therefore looking at the distribution of the PV in the ocean can be used to determine where certain waters have originated and what processes they have undergone on their journey.

A nice example showing this utility of the PV is in observation collected at the Gulf Stream during the CLIMODE experiment.

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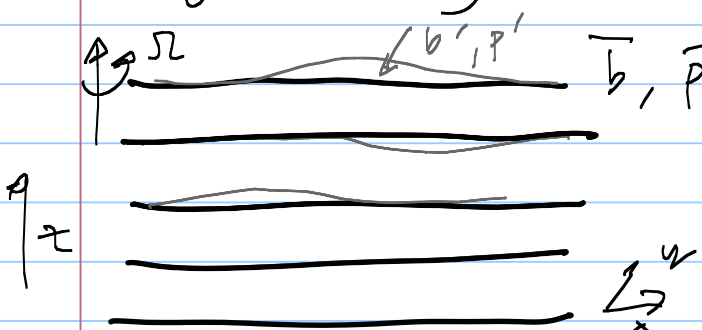
While the conservative nature of the PV gives it properties similar to a dye, unlike dye it is not a passive tracer (i.e. a tracer that does not affect the dynamics of the flow) because the spatial structure of the PV field affects the flow field as a consequence of PV INVERTIBILITY.

Let's look at a concrete example of this.

Consider a flow with a constant background stratification, constant Coriolis parameter that is perturbed very slightly so that:



$$b = \bar{b} + b'$$
$$p = \bar{p} + p'$$
$$u = u', \quad v = v'$$

b' small compared to $\bar{b}$

u', v' small $\rightarrow$ low Rossby number

$p' \rightarrow$ pressure hydrostatic $\sqrt{}\ (H/L) \ll 1$

velocities geostrophic

The Ertel PV for this system is:

$$q = f\bar{N}^2 + f\frac{\partial b'}{\partial z} + \zeta \bar{N}^2 + \zeta\frac{\partial b'}{\partial z} + \frac{\partial u'}{\partial z}\frac{\partial b'}{\partial y} - \frac{\partial v'}{\partial z}\frac{\partial b'}{\partial x}$$

These are all higher order terms and can be neglected

$$q = \bar{q} + q' \simeq f\bar{N}^2 + f\frac{\partial b'}{\partial z} + \bar{N}^2 \zeta$$

$$\bar{q} = f\bar{N}^2 \qquad q' = f\frac{\partial b'}{\partial z} + \bar{N}^2\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right)$$

From the geostrophic balance:

$$u' = -\frac{1}{\rho_0 f}\frac{\partial p'}{\partial y} \qquad v' = \frac{1}{\rho_0 f}\frac{\partial p'}{\partial x}$$

We can introduce a streamfunction

$$\psi = \frac{1}{\rho_0 f} p' \qquad p' = \rho_0 f \psi$$

From the hydrostatic relation:

$$0 = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + b'$$

$$\Rightarrow \quad b' = f\frac{\partial \psi}{\partial z}$$

and thus the PV perturbation becomes

$$q' = f^2\frac{\partial^2 \psi}{\partial z^2} + \bar{N}^2 \nabla_h^2 \psi$$

If we define a quantity $\hat{q}' = q'/\bar{N}^2$

$$\hat{q}' = \nabla_h^2 \psi + \frac{f^2}{\bar{N}^2}\frac{\partial^2 \psi}{\partial z^2}$$

This quantity has units of vorticity and is known as the Quasi-geostrophic PV for a continuously stratified fluid.

Thus given a distribution of the PV field, this equation can be solved to obtain the streamfunction which can be used to calculate the geostrophic flow AND the buoyancy perturbation:

$$q' \xrightarrow[\text{INVERSION}]{\text{PV}} \psi \xrightarrow[\text{hydrostatic balance}]{\text{Geostrophic}} u', v', b'$$

⇒ From one single scalar variable we can obtain all the dynamically important variables u', v', b'.

Thus the PV is far more than just a tracer for tagging fluid parcels, it is a DYNAMICAL TRACER that yields information on all flow quantities of interest.

Lets take a closer look at the QG PV

$$\hat{q}' = \nabla_h^2 \psi + \frac{f^2}{\bar{N}^2}\frac{\partial^2 \psi}{\partial z^2}$$

Notice that if we define a stretched z-coordinate:

$$z_* = \frac{\bar{N}^2}{f^2} z$$

and a new coordinate system, with coordinates

$$(x_*, y_*, z_*) = \left(x, y, \frac{\bar{N}^2}{f^2} z\right)$$

Then $$\frac{f^2}{\bar{N}^2}\frac{\partial^2 \psi}{\partial z^2} = \frac{\partial^2 \psi}{\partial z_*^2}$$

So that the QG PV becomes:

$$\nabla_*^2 \psi = \hat{q}'$$

Where $$\nabla_*^2 = \frac{\partial^2}{\partial x_*^2} + \frac{\partial^2}{\partial y_*^2} + \frac{\partial^2}{\partial z_*^2}$$

is the 3-D Laplacian in the new coordinate system

What do solutions of this equation look like?

This equation is known as Poisson's equation. It is the same equation used to solve for the gravitational potential induced by a distribution of masses:

$$\underset{\substack{\uparrow \\ \text{gravitational} \\ \text{potential}}}{\nabla^2 \phi} = 4\pi \underset{\substack{\uparrow \\ \text{gravitational} \\ \text{constant}}}{G} \rho \leftarrow \text{mass density}$$

We can use our intuition and knowledge from classical mechanics of solutions to this equation to infer key characteristics of the solutions to the QG PV equation.

For example, for a point mass the gravitational potential is:

$$\phi = -\frac{Gm}{|\vec{r}\,' - \vec{r}|}$$

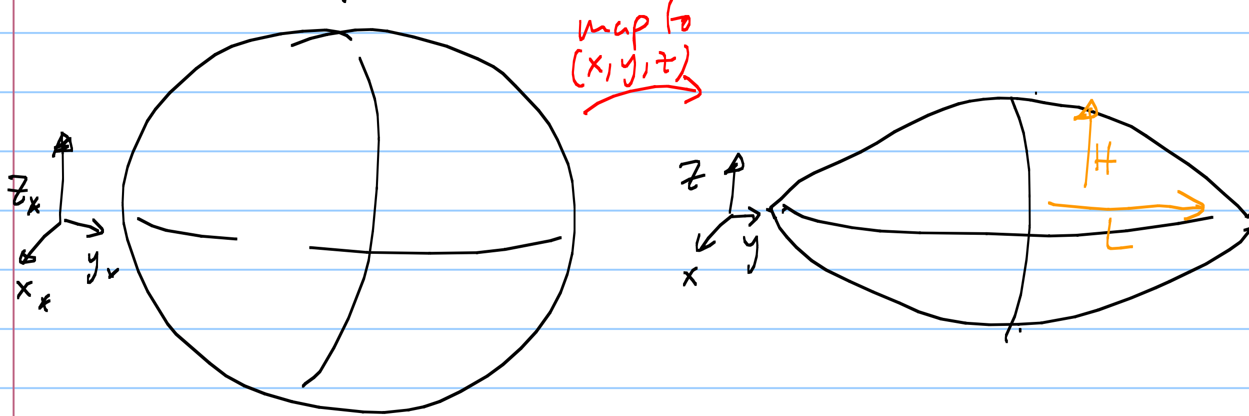
where $\vec{r}\,'$ is the location of the mass

So surfaces of constant ϕ are spheres with a center at the point mass.

We can use this result to figure out the streamfunction associated with a point source distribution of PV:

$$\nabla_x^2 \psi = \hat{q}_p' \leftarrow \text{point source}$$

In x_*, y_*, z_* surfaces of constant Ψ will be spheres



While in (x,y,z) space Ψ will take the shape of an ellipsoid with an aspect ratio

$$\frac{H}{L} \sim \frac{f}{N} < 1$$

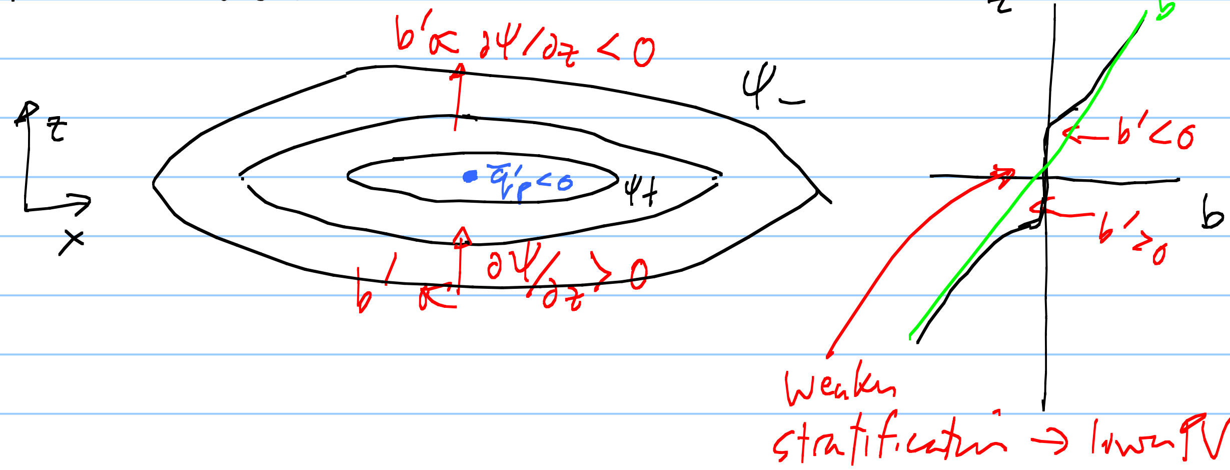
in the ocean

This natural selection of the flow having an aspect ratio of f/N is a reflection of the constraints that stratification and rotation impose on the fluid recall that

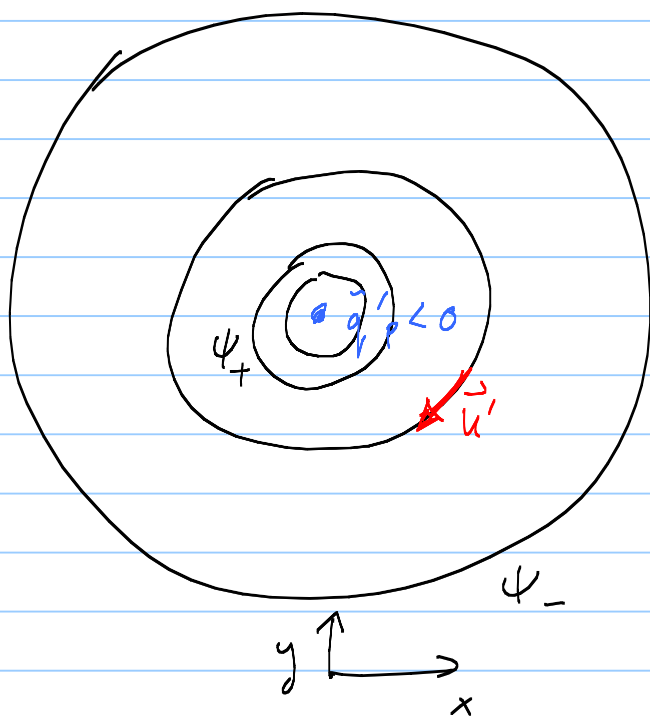
- Rotation tends to make flows vertically homogeneous (remember the Taylor Proudman effect)
- Stratification tends to keep flow flat

→ The relative strengths of these two effects goes as f/N and thus selects the aspect ratio of the flow.

Imagine that we had a negative point source of PV $\tilde{q}'_p < 0$ then the stream function field would be:



Looking down from above:



→ anticyclonic flow associated with low PV anomaly